

Quantum dynamics in a time-dependent cylindrical trap

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Solutions of the Schrödinger equation is examined for a particle inside a cylindrical trap of a circular time-dependent cross-section: the analytical expressions for the energy and momentum expectation values are derived with respect to the exact solutions, and the adiabatic and the sudden change of the boundary are discussed. The density profile in the time, in a given observation point, resembles the diffraction-in-time pattern observed in a suddenly released particle but, with an enhanced fringe visibility. Numerical computations are presented for both contracting and expanding boxes.

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I. INTRODUCTION

Quantum mechanical particle motions which are subject to time-dependent boundary conditions belong to a main class of the time-dependent dynamical systems. Finding the exact solutions of the Schrödinger equation in these time-dependent systems, is not an easy task [1–5].

Diffraction in time [1], quantum temporal oscillations of matter waves released from a confining region, is just one of the very interesting effects that are seen in such systems. For a beam of particles which is incident from the left on a moving wall the visibility of the fringes is enhanced in comparison with the motionless case [6].

Moshinsky's theoretical work has been studied for particles which are suddenly released from a 1D box [7], from a spherical [8] and from a cylindrical trap [9].

Exact solutions of the Schrödinger equation for a particle in a 1D box with a moving wall has been found [2, 3]. Using the semiclassical approximation, Luz and Cheng [10] evaluated the exact propagator of the problem.

It has been noted by Godoy and Okamura [9], that particles motion in a nanoscopic circuit can be regarded as a quantum wave moving inside a wave guide filled with a scattering medium, and transient currents in these circuits can be produced by rapidly removing (or adding) boundary walls to the guide. See [11] for a recent review on quantum transients.

With this in mind, we want to examine the solutions of the Schrödinger equation for a particle inside a circle with a varying radius. There are two extra motivations for this study. First, the sudden removal of the boundary is an idealized case. Second, possible applications to optical effects connected with the moving mirrors are additionally encouraging. Although, the limit of infinite velocity of the moving boundary is clearly corresponds to the sudden removal of the wall, due to the non-relativistic nature of the Schrödinger equation, the speed of the wall has to be very less than the speed of light, $u \ll c$.

II. EXACT SOLUTION

Consider a particle with mass μ inside a cylindrical wave guide of circular cross-section. The cylinder has a radius $L(t) = a + ut$, which uniformly changes with time, and its axis is taken along the z -direction. The potential energy function is zero if $r < L(t)$ and infinite otherwise. The 2D Schrödinger equation for the transverse wavefunction in the polar coordinates $\mathbf{r} = (\rho, \phi)$ is then

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2\mu} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \Psi(\mathbf{r}, t). \quad (1)$$

In the absence of the boundary motion, $u = 0$, the solutions of eq.(1) would be

$$u_{mn}(\mathbf{r}) = \frac{\sqrt{2}}{a} \frac{1}{|J_{m+1}(x_{mn})|} J_m \left(x_{mn} \frac{\rho}{a} \right) \frac{e^{im\phi}}{\sqrt{2\pi}}, \quad (2)$$

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with eigenvalues $E_{mn} = \hbar^2 x_{mn}^2 / (2\mu a^2)$, where x_{mn} are the roots of Bessel functions, $J_m(x_{mn}) = 0$ with $m = 0, 1, 2, \dots$ and $n = 1, 2, 3, \dots$. All Bessel functions with $m \neq 0$ have a zero at the origin, but one has not to consider these zeros to have a non-zero wavefunction. Thus, the *instantaneous* energy eigenfunctions and eigenvalues would, respectively, be

$$u_{mn}(\mathbf{r}, t) = \frac{\sqrt{2}}{L(t)} \frac{1}{|J_{m+1}(x_{mn})|} J_m\left(x_{mn} \frac{\rho}{L(t)}\right) \frac{e^{im\phi}}{\sqrt{2\pi}} e^{ikz}, \quad (3)$$

$$E_{mn}(t) = \frac{\hbar^2}{2\mu} \left(k^2 + \frac{x_{mn}^2}{L^2(t)} \right), \quad (4)$$

which satisfy the relation

$$i\hbar \frac{\partial}{\partial t} \left(\exp \left[-i \frac{1}{\hbar} \int_0^t dt' E_{mn}(t') \right] u_{mn}(\mathbf{r}, t) \right) = E_{mn}(t) \left(\exp \left[-i \frac{1}{\hbar} \int_0^t dt' E_{mn}(t') \right] u_{mn;k}(\mathbf{r}, t) \right). \quad (5)$$

Exact solutions of the Schrödinger equation (1) for the above problem have been found [12],

$$\Psi_{mn}(\rho, \phi, t) = C \frac{1}{L(t)} \exp \left[\frac{i\mu}{2\hbar} u \frac{\rho^2}{L(t)} - i \frac{\hbar}{2\mu} x_{mn}^2 \frac{t}{aL(t)} \right] J_m\left(x_{mn} \frac{\rho}{L(t)}\right) \frac{e^{im\phi}}{\sqrt{2\pi}}. \quad (6)$$

Unknown coefficient C is determined by the normalization condition $\int_0^{L(t)} d\rho \int d\phi |\Psi_{mn}(\mathbf{r}, t)|^2 = 1$, which apart from a constant phase factor yields in $C = \sqrt{2}/J_{m+1}(x_{mn})$, by using the orthogonality of the Bessel functions [13]

$$\int_0^1 ds s J_m(x_{mp}s) J_m(x_{mq}s) = \frac{1}{2} [J_{m+1}(x_{mp})]^2 \delta_{pq}. \quad (7)$$

Thus, exact solutions read

$$\Psi_{mn}(\rho, \phi, t) = \exp \left[i\alpha \xi(t) \left(\frac{\rho}{L(t)} \right)^2 - ix_{mn}^2 \frac{1 - 1/\xi(t)}{4\alpha} \right] u_{mn}(\rho, \phi, t), \quad (8)$$

where we have introduced dimensionless quantities $\alpha = \mu a u / (2\hbar)$ and $\xi(t) = L(t)/a$. It must be mentioned that including the motion in the z -direction, would introduce an extra factor $\exp[ikz - i\hbar k^2 t / (2\mu)]$. Functions $\Psi_{mn}(\mathbf{r}, t)$ vanish at $\rho = L(t)$, remain normalized as the boundary moves and form a complete orthogonal set. Thus, the general solution of eq. (1) can be expanded in terms of functions (8),

$$\Psi(\mathbf{r}, t) = \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} c_{m'n'} \Psi_{m'n'}(\mathbf{r}, t), \quad (9)$$

with time-independent coefficients $c_{m'n'}$ determined from the relation

$$c_{m'n'} = \int_0^a d\rho \rho \int_0^{2\pi} d\phi \Psi_{m'n'}^*(\mathbf{r}, 0) \Psi(\mathbf{r}, 0). \quad (10)$$

General solution can also be written as a superposition of instantaneous eigenfunctions,

$$\Psi(\mathbf{r}, t) = \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} d_{m'n'}(t) \exp \left[-\frac{i}{\hbar} \int_0^t dt' E_{m'n'}(t') \right] u_{m'n'}(\mathbf{r}, t) \equiv \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} b_{m'n'}(t) u_{m'n'}(\mathbf{r}, t), \quad (11)$$

now with time-dependent coefficients $b_{m'n'}(t)$ determined from the relation

$$b_{m'n'}(t) = \int_0^{L(t)} d\rho \rho \int_0^{2\pi} d\phi u_{m'n'}^*(\mathbf{r}, t) \Psi(\mathbf{r}, t). \quad (12)$$

Using eqs. (9) and (12) and the orthonormality of functions $\exp(im\phi)/\sqrt{2\pi}$, one obtains

$$b_{m'n'}(t) = \frac{2}{|J_{m'+1}(x_{m'n'})|} \sum_{n''=1}^{\infty} c_{m'n''} \frac{1}{|J_{m'+1}(x_{m'n''})|} \exp \left[-ix_{m'n''}^2 \frac{1 - 1/\xi(t)}{4\alpha} \right] I_{m'n'n''}^*(t, \alpha), \quad (13)$$

where

$$I_{m'n'n''}(t, \alpha) = \int_0^1 ds s e^{-i\alpha\xi(t)s^2} J_{m'}(x_{m'n'}s) J_{m'}(x_{m'n''}s) , \quad (14)$$

with the property $I_{mnn'}(t, -\alpha) = I_{mnn'}^*(t, \alpha)$.

If the particle is initially in an energy eigenstate, *i.e.*, $\Psi(\mathbf{r}, 0) = u_{mn}(\mathbf{r}, 0)$, then

$$c_{m'n'} = \delta_{mm'} \frac{2}{|J_{m+1}(x_{mn}) J_{m+1}(x_{mn'})|} I_{mnn'}(0, \alpha) , \quad (15)$$

which is not an unexpected result as the angular momentum operator $L_z = -i\hbar \partial/\partial\phi$ commutes with the Hamiltonian of the system. For $\alpha \ll 1$, one has $I_{mnn'}(0, \alpha) \simeq \delta_{nn'} (J_{m+1}(x_{mn}))^2/2$, and thus from eq. (9), one obtains $\Psi(\mathbf{r}, t) \simeq \Psi_{mn}(\mathbf{r}, t)$, which is in agreement with the result of the adiabatic approximation.

For completeness we should add the relations of the energy expectation value and the propagator of the problem. The expectation value of the energy of the particle is obtained from

$$\begin{aligned} \langle H \rangle(t) &= \int_0^{L(t)} d\rho \rho \int_0^{2\pi} d\phi \Psi^*(\mathbf{r}, t) i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) \\ &= \sum_{m'n'} |d_{m'n'}(t)|^2 E_{m'n'}(t) = \sum_{m'n'} |b_{m'n'}(t)|^2 E_{m'n'}(t) . \end{aligned} \quad (16)$$

Propagator of the problem can be constructed as follows:

$$\begin{aligned} |\Psi(t)\rangle &= S(t, t') |\Psi(t')\rangle = \sum_{mn} \sum_{m'n'} |\Psi_{mn}(t)\rangle \langle \Psi_{mn}(t) | S(t, t') | \Psi_{m'n'}(t') \rangle \langle \Psi_{m'n'}(t') | \Psi(t') \rangle \\ &= \sum_{mn} |\Psi_{mn}(t)\rangle \langle \Psi_{mn}(t') | \Psi(t') \rangle , \end{aligned}$$

where $S(t, t')$ is the time evolution operator and we have used the fact that if the particle is in the state $|\Psi_{mn}\rangle$ at t' , it remains in that state as the wall moves, *i.e.*, $S(t, t') |\Psi_{mn}(t')\rangle = |\Psi_{mn}(t)\rangle$. Now, we write this equation in the form

$$\Psi(\mathbf{r}, t) = \int_0^a d\rho' \rho' \int_0^{2\pi} d\phi' K(\mathbf{r}, t; \mathbf{r}', t') \Psi(\mathbf{r}', t') , \quad (17)$$

where we have introduced the propagator as

$$\begin{aligned} K(\mathbf{r}, t; \mathbf{r}', t') &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Psi_{mn}(\mathbf{r}, t) \Psi_{mn}^*(\mathbf{r}', t') \\ &= \frac{2}{L(t)L(t')} \sum_{mn} \frac{1}{[J_{m+1}(x_{mn})]^2} \\ &\quad \times \exp \left[\frac{i\mu u}{2\hbar} \left(\frac{\rho^2}{L(t)} - \frac{\rho'^2}{L(t')} \right) - \frac{i\hbar}{2\mu} \frac{x_{mn}^2}{a} \left(\frac{t}{L(t)} - \frac{t'}{L(t')} \right) \right] \\ &\quad \times J_m \left(x_{mn} \frac{\rho}{L(t)} \right) J_m \left(x_{mn} \frac{\rho'}{L(t')} \right) \times \frac{e^{im\phi}}{\sqrt{2\pi}} \frac{e^{-im\phi'}}{\sqrt{2\pi}} . \end{aligned} \quad (18)$$

Expectation value of an arbitrary operator \hat{A} is given by

$$\langle \Psi | \hat{A} | \Psi \rangle(t) = \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{m'n'}^* c_{mn} \langle \Psi_{m'n'}(t) | \hat{A} | \Psi_{mn}(t) \rangle , \quad (19)$$

where

$$\langle \Psi_{m'n'}(t) | \hat{A} | \Psi_{mn}(t) \rangle = \int_0^{2\pi} d\phi \int_0^{L(t)} d\rho \rho \Psi_{m'n'}^*(\rho, \phi, t) A(\rho, \phi) \Psi_{mn}(\rho, \phi, t) , \quad (20)$$

are the matrix elements of the operator with respect to the states $\Psi_{mn}(\mathbf{r}, t)$. It must be noted that the expectation values can also be found by expanding the wavefunction in terms of the instantaneous energy eigenfunctions as we did

in eq. (16). Our aim is the examination of the uncertainty relation. To this end, we at first compute matrix elements of the position, the momentum and the Hamiltonian operators.

The position and the momentum operators in the arbitrary radial direction $\hat{\rho}_0 = \hat{x} \cos \phi_0 + \hat{y} \sin \phi_0$, are

$$\begin{aligned} q_0 &= x \cos \phi_0 + y \sin \phi_0, \\ p_0 &= p_x \cos \phi_0 + p_y \sin \phi_0 = \frac{\hbar}{i} \left[\cos \phi_0 \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \phi \frac{\partial}{\partial \phi} \right) + \sin \phi_0 \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \phi \frac{\partial}{\partial \phi} \right) \right], \end{aligned} \quad (21)$$

which are canonical conjugate variables, $[q_0, p_0] = i\hbar$. By straightforward algebra, one obtains

$$\langle \Psi_{m'n'}(t) | q_0 | \Psi_{mn}(t) \rangle = 0, \quad (22)$$

$$\langle \Psi_{m'n'}(t) | p_0 | \Psi_{mn}(t) \rangle = 0, \quad (23)$$

$$\langle \Psi_{m'n'}(t) | q_0^2 | \Psi_{mn}(t) \rangle = \delta_{m,m'} \exp \left[i(x_{mn'}^2 - x_{mn}^2) \frac{1 - 1/\xi(t)}{4\alpha} \right] a^2 \xi^2(t) \frac{2\pi A_{mn'n}^{(3)}}{|J_{m+1}(x_{mn'}) J_{m+1}(x_{mn})|}, \quad (24)$$

$$\begin{aligned} \langle \Psi_{m'n'}(t) | p_0^2 | \Psi_{mn}(t) \rangle &= \delta_{m,m'} \exp \left[i(x_{mn'}^2 - x_{mn}^2) \frac{1 - 1/\xi(t)}{4\alpha} \right] \frac{\hbar^2}{a^2} \frac{2\pi}{|J_{m+1}(x_{mn'}) J_{m+1}(x_{mn})|} \\ &\times \left[4\alpha^2 A_{mn'n}^{(3)} + \frac{1}{\xi^2(t)} (m^2 A_{mn'n}^{(-1)} - B_{mn'n}^{(0)} - C_{mn'n}^{(1)} - i \frac{4\alpha}{\xi(t)} (A_{mn'n}^{(1)} + B_{mn'n}^{(2)})) \right], \end{aligned} \quad (25)$$

$$\begin{aligned} \langle \Psi_{m'n'}(t) | H | \Psi_{mn}(t) \rangle &= -\frac{\hbar^2}{2\mu} \int_0^{2\pi} \int_0^{L(t)} d\rho \rho \Psi_{m'n'}^*(\rho, \phi, t) \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial \phi^2} \right] \Psi_{mn}(\rho, \phi, t) \\ &= \frac{1}{\pi} \frac{\langle \Psi_{m'n'}(t) | p_0^2 | \Psi_{mn}(t) \rangle}{2\mu}, \end{aligned} \quad (26)$$

where

$$A_{mn'n}^{(k)} = \int_0^1 ds s^k J_m(x_{mn'} s) J_m(x_{mn} s), \quad (27)$$

$$B_{mn'n}^{(k)} = \int_0^1 ds s^k J_m(x_{mn'} s) \frac{d}{ds} J_m(x_{mn} s), \quad (28)$$

$$C_{mn'n}^{(k)} = \int_0^1 ds s^k J_m(x_{mn'} s) \frac{d^2}{ds^2} J_m(x_{mn} s), \quad (29)$$

are constant coefficients. As one sees, all of the expectation values are independent of ϕ_0 , the direction we started with it. $A_{mn'n}^{(k)}$ is symmetric with respect to two indices n' and n , while $B_{mn'n}^{(k)}$ and $C_{mn'n}^{(k)}$ do not have any symmetry. It must be mentioned that $A_{mn'n}^{(-1)}$ diverges for $m = 0$, but it is not problematic because the wave-function is not ϕ -dependent for $m = 0$ and thus, the term $m^2 A_{mn'n}^{(-1)}$ will not appear in such a case.

From eqs. (19), (22) and (23), one sees that position and momentum expectation values are zero, irrespective of the shape of the wavefunction, $\langle \Psi | q_0 | \Psi \rangle(t) = 0$ and $\langle \Psi | p_0 | \Psi \rangle(t) = 0$; and from eqs. (19) and (26), $\langle \Psi | H | \Psi \rangle(t) = \langle \Psi | p_0^2 | \Psi \rangle(t) / (2\pi\mu)$.

III. UNCERTAINTY RELATIONS

Let us suppose that one succeed to construct the initial wavefunction to be the state $\Psi_{mn}(\mathbf{r}, 0)$. Such a task can be done by appropriately superposing instantaneous energy eigenstates. If the particle is initially in the state $\Psi(\mathbf{r}, 0) = \Psi_{mn}(\mathbf{r}, 0)$, then

$$c_{m'n'} = \delta_{mm'} \delta_{nn'}, \quad (30)$$

$$b_{m'n'}(t) = \delta_{mm'} \frac{2}{|J_{m+1}(x_{mn}) J_{m+1}(x_{mn'})|} \exp \left[-i x_{mn}^2 \frac{1 - 1/\xi(t)}{4\alpha} \right] I_{mnn'}^*(t, \alpha). \quad (31)$$

From eqs. (9) and (30), one finds that if $\Psi(\mathbf{r}, 0) = \Psi_{mn}(\mathbf{r}, 0)$, then time-evolved wavefunction would be $\Psi_{mn}(\mathbf{r}, t)$ in any instant of time.

By using eqs. (16) and (31), one obtains

$$\frac{\langle H \rangle_{mn}(t)}{\langle H \rangle_{mn}(0)} = \left(\frac{a}{L(t)} \right)^2 \frac{\sum_{n'} \left(\frac{x_{mn'}}{J_{m+1}(x_{mn'})} \right)^2 |I_{mnn'}(t, \alpha)|^2}{\sum_{n'} \left(\frac{x_{mn'}}{J_{m+1}(x_{mn'})} \right)^2 |I_{mnn'}(0, \alpha)|^2} \quad (32)$$

for the energy expectation value. From this relation it is not obvious how the energy expectation value changes with time, whereas eq. (34) is completely informative in this connection.

Using eqs.(22), (23), (24) and (25), one obtains

$$\langle q_0^2 \rangle_{mn}(t) = a^2 \xi^2(t) \frac{2\pi A_{mnn}^{(3)}}{|J_{m+1}(x_{mn})|^2}, \quad (33)$$

$$\langle p_0^2 \rangle_{mn}(t) = \frac{\hbar^2}{a^2} \frac{2\pi}{|J_{m+1}(x_{mn})|^2} \left[4\alpha^2 A_{mnn}^{(3)} + \frac{1}{\xi^2(t)} (m^2 A_{mnn}^{(-1)} - C_{mnn}^{(1)}) \right] \equiv (2\pi\mu) \langle H \rangle_{mn}(t), \quad (34)$$

$$\Delta q_{0,mn}(t) \equiv \sqrt{\langle q_0^2 \rangle_{mn}(t) - \langle q_0 \rangle_{mn}^2(t)} = a \xi(t) \frac{\sqrt{2\pi A_{mnn}^{(3)}}}{|J_{m+1}(x_{mn})|}, \quad (35)$$

$$\Delta p_{0,mn}(t) \equiv \sqrt{\langle p_0^2 \rangle_{mn}(t) - \langle p_0 \rangle_{mn}^2(t)} = \frac{\hbar}{a} \sqrt{4\alpha^2 \left(\frac{\Delta q_{0,mn}(t)}{L(t)} \right)^2 + \frac{1}{\xi^2(t)} \frac{2\pi(m^2 A_{mnn}^{(-1)} - C_{mnn}^{(1)})}{|J_{m+1}(x_{mn})|^2}}, \quad (36)$$

$$\Delta q_{0,mn}(t) \Delta p_{0,mn}(t) = \frac{\hbar}{2} \frac{4\pi}{|J_{m+1}(x_{mn})|^2} \sqrt{(m^2 A_{mnn}^{(-1)} - C_{mnn}^{(1)}) A_{mnn}^{(3)} + (2\alpha A_{mnn}^{(3)})^2 \xi^2(t)} \quad (37)$$

where, for abbreviation expectation values have been shown as $\langle \dots \rangle_{mn}(t)$ instead of $\langle \Psi_{mn}(t) | \dots | \Psi_{mn}(t) \rangle$; and $B_{mnn}^{(0)} = 0$. In the appendix the solutions of $A_{mnn}^{(k)}$ and $C_{mnn}^{(1)}$ will be given in terms of the hypergeometric and the regularized hypergeometric functions.

Considering the above equation, one finds

- i) $\Delta q_{0,mn}(t)$ changes linearly with time, it increases for expansion while decreases for contraction. This is an expected result.
- ii) There is an additional time- and \hbar -independent contribution to $\Delta p_{0,mn}$ in comparison with the stationary boundary. The second term under the radical sign is time-dependent and decreases with expansion while increases in contraction. For an expanding circle, $\Delta p_{0,mn}(t)$ becomes constant in the limit $t \rightarrow \infty$.
- iii) Eq. (37) shows that the product of the uncertainties is time-dependent, a result which is absent in stationary systems. According to the general uncertainty principle $\Delta q_i \Delta p_i \geq \hbar/2$, where q_i and p_i are the non-commuting i -component of the conjugate variables \mathbf{r} and \mathbf{p} . This means the quantity $4\pi \sqrt{(m^2 A_{mnn}^{(-1)} - C_{mnn}^{(1)}) A_{mnn}^{(3)}} / |J_{m+1}(x_{mn})|^2$, which is the product of uncertainties in the corresponding stationary system, must be greater than 1.

IV. DISCUSSION

As eq.(34) shows and fig. 1 confirms, expectation value of the energy decreases with time for an expanding box while increases for a contracting one, as time elapses. This is an understandable result in the context of the old quantum theory [15], as is reasonable by considering the uncertainty relations (35) and (36) [16].

Fig. 2 shows that, in the presented region, $A_{mnn}^{(-1)}/|J_{m+1}(x_{mn})|^2$ and $A_{mnn}^{(3)}/|J_{m+1}(x_{mn})|^2$ becomes constant for $n \geq 10$, while $|C_{mnn}^{(1)}|/|J_{m+1}(x_{mn})|^2$ increases with n . In addition, $A_{mnn}^{(3)}/|J_{m+1}(x_{mn})|^2$ is approximately independent of quantum number m for $n \geq 10$, while $C_{mnn}^{(1)}/|J_{m+1}(x_{mn})|^2$ is approximately the same for all values of n and dominates for $n \geq 3$. From these considerations; and eqs. (33) and (34), one finds that at a given time and a given wall velocity, $\langle q_0^2 \rangle_{mn}$ becomes independent of n and m for $n \geq 10$, while $\langle p_0^2 \rangle_{mn}$ is independent of m , but increases with n .

For numerical calculations of the wavefunction, we define two new quantities $\lambda_{mn} = 2\pi a/x_{mn}$ and $\nu_{mn} = E_{mn}/(2\pi\hbar)$ and from them new dimensionless position and time coordinates $\eta = \rho/\lambda_{mn}$ and $T = \nu_{mn}t$; and dimensionless radial

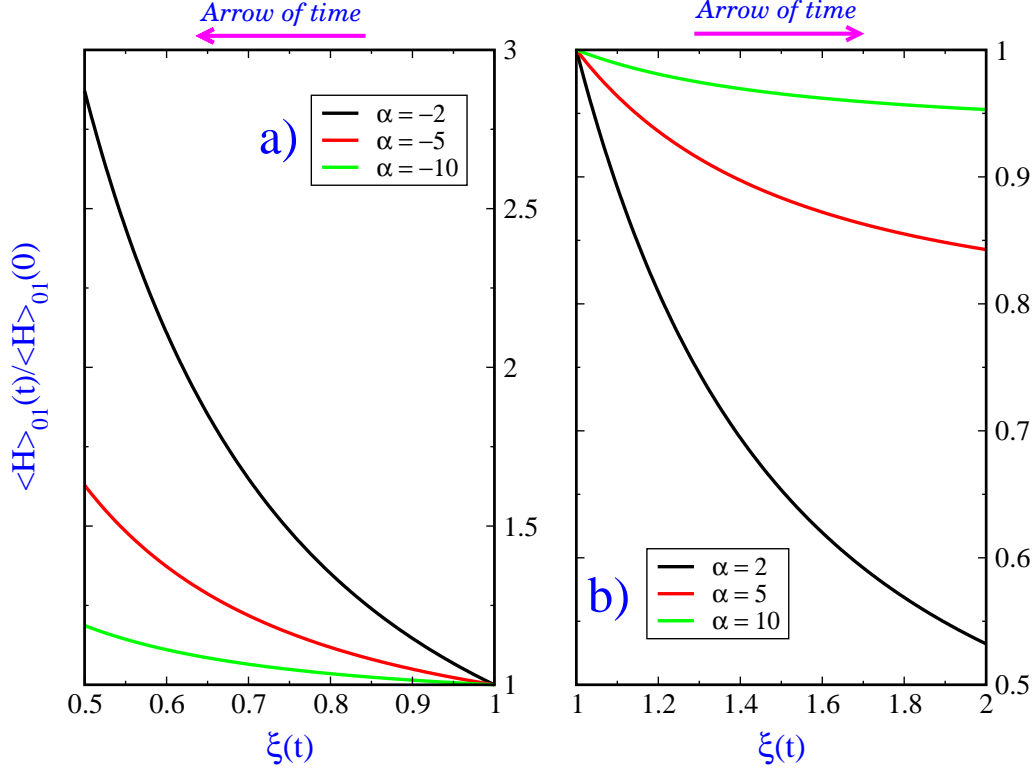


FIG. 1: (Color online) $\langle H \rangle_{01}(t)/\langle H \rangle_{01}(0)$ versus $\xi(t)$, for (a) contracting and b) expanding circular boxes. Expectation value of the energy of the confined particle increases (decreases) for a contracting (an expanding) box.

probability density $\varrho_{mn}(\eta, T) = \lambda_{mn}^2 \eta |R(\eta, T)|^2$, where $R(\eta, T)$ stands for the radial part of the wavefunction. With these quantities, the location of the wall is determined from the relation

$$\xi(T) = 1 + 2\pi \frac{\alpha}{\alpha_{mn}^2} T,$$

where,

$$\alpha_{mn} = \frac{\mu a}{2\hbar} v_{mn} = \frac{\mu a}{2\hbar} \frac{\hbar x_{mn}}{\mu a} = \frac{x_{mn}}{2},$$

corresponds to the wall velocity that is equal to the initial classical velocity of the confined particle.

We have plotted $\varrho_{mn}(\eta, T)$ in fig. 3 for a particle initially in the state $u_{0,10}$, against η at different times.

A system undergoes an adiabatic evolution when $t_i \ll t_e$, where t_i (t_e) is the time-scale over which internal (external) variables of the system changes [17]. For our particle-in-a-box, $t_e = a/u$ and $t_i = a/v_{mn}$. Figures 4 and 5 show that the particle, initially in an instantaneous energy eigenstate will remain in the same eigenstate when $|\alpha| \ll \alpha_{mn}$, as a consequence of the adiabatic approximation. Figure 4 shows that sudden approximation does not work for a contracting well [15].

Figure 6 represents density probability versus time in a given observation point.

T_1 and T_2 are dimensionless classical flight times from the front and back edges of the circle to the dimensionless observation point. In contrast to classical mechanics, when $\alpha > \alpha_{mn}$, one sees a non-monotonous increasing behavior of the density for $T < T_1$. A non-monotonous decreasing behavior is seen for $T > T_2$, irrespective of the wall velocity. Difference between the adjacent extremums (visibility) decreases with α . The constructive interference with the reflected components from the moving boundary for $\alpha < \alpha_{mn}$, increases the fringe visibility. At long times, the

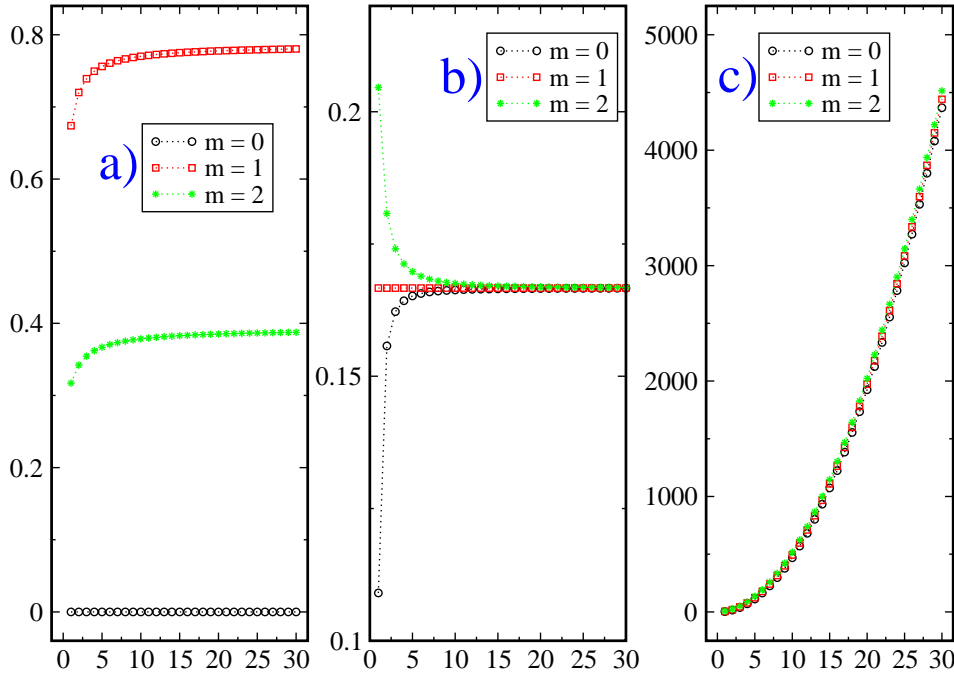


FIG. 2: (Color online) (a) $A_{mnn}^{(-1)}/|J_{m+1}(x_{mn})|^2$, (b) $A_{mnn}^{(3)}/|J_{m+1}(x_{mn})|^2$ and (c) $|C_{mnn}^{(1)}|/|J_{m+1}(x_{mn})|^2$ versus quantum number n for different values of quantum number m .

behaviour of the density in the observation point, is the same for all values of the velocity parameter α . This can be understood by considering the behavior of functions Ψ_{mn} at long times: at this limit at an arbitrary observation point $\mathbf{r} = \mathbf{r}_0$, the coordinate-deepened phase goes to zero. Thus, from eqs. (9) and (10), one obtains

$$\lim_{t \rightarrow \infty} |\psi(\mathbf{r}_0, t)| = |u_{mn}\mathbf{r}_0, t)| \Big|_{t \rightarrow \infty} \simeq \frac{2}{ut} \frac{1}{|J_{m+1}(x_{mn})|^2} J_m(x_{mn} \frac{\rho}{ut}) \Big|_{t \rightarrow \infty}, \quad (38)$$

which ultimately becomes zero.

V. SUMMARY

Exact solutions of the Schrödinger equation for a particle in a circular hard box with a moving wall in uniform motion, contain a coordinate-dependent phase $\exp\left[i\frac{\mu u}{2\hbar} \frac{\rho^2}{L(t)}\right]$, which is absent in the corresponding stationary boundary problem, whose significance has been already emphasised [4, 14] in 1D systems. Propagator of the problem is constructed and the matrix elements of the position, the momentum and the energy observables were derived with respect to the exact solutions. It was seen that the uncertainty product increases with time due to a time- and velocity-dependent term; and expectation value of the energy increases (decreases) with time, for a contracting (an expanding) boxes which is consistent with the uncertainty relations. The density profile in the time, in a given location, resembles the diffraction-in-time pattern observed in a suddenly released particle, but it actually shows an enhancement of the visibility of the fringes, the 2D version of the EDIT (Enhanced diffraction in time) reported in 1D in [6]. We have provided a full characterization of the quantum transients in an expanding/contracting cylindrical

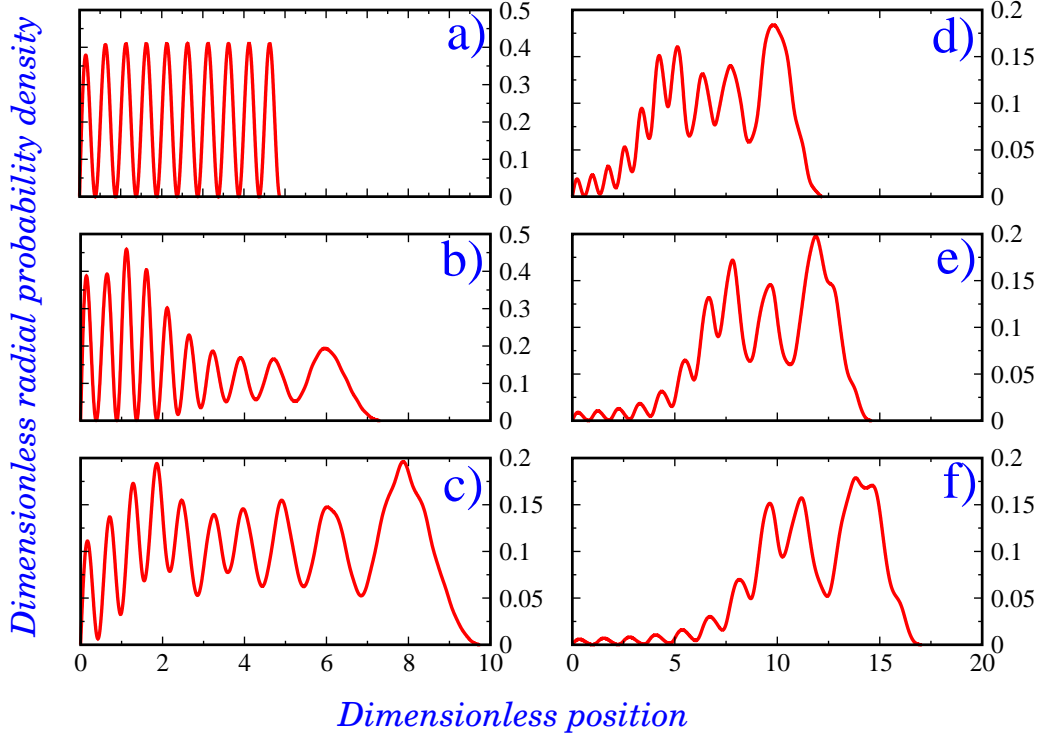


FIG. 3: (Color online) Dimensionless radial probability density $\varrho_{0,10}(\eta, T)$ for $\alpha = \alpha_{0,10}$ versus dimensionless position coordinate η at the instant of time when the wall arrives at (a) $\xi = 1$, (b) $\xi = 1.5$, (c) $\xi = 2$, (d) $\xi = 2.5$, (e) $\xi = 3$ and (f) $\xi = 3.5$.

box resulting from the breakdown of adiabaticity. We close by noticing that, should it be desirable, it is possible to suppress quantum transients in this system by engineering a shortcut to adiabaticity [18].

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VI. APPENDIX

Here, we write integrals $A_{mnn}^{(-1)}$, $A_{mnn}^{(3)}$ and $C_{mnn}^{(1)}$ in terms of generalized and regularized generalized hypergeometric functions:

$$\begin{aligned}
 A_{mnn}^{(-1)} &= \frac{1}{2m} \int_0^1 ds J_m(x_{mn}s) (J_{m+1}(x_{mn}s) + J_{m-1}(x_{mn}s)) \\
 &= 4^{-m} (2m-1)! {}_2\bar{F}_3 \left(m, m + \frac{1}{2}; m+1, m+1, 2m+1; -(x_{mn})^2 \right) \quad \text{if } m \neq 0, \quad (39)
 \end{aligned}$$

$$A_{mnn}^{(3)} = \begin{cases} 4^{-m} m(m+1)(x_{mn})^2 (2m-1)! {}_2\bar{F}_3 \left(m + \frac{1}{2}, m+2; m+1, m+3, 2m+1; -(x_{mn})^2 \right) & \text{if } m \neq 0, \\ \frac{1}{4} {}_2F_3 \left(\frac{1}{2}, 2; 1, 1, 3; -(x_{0n})^2 \right) & \text{if } m = 0, \end{cases} \quad (40)$$

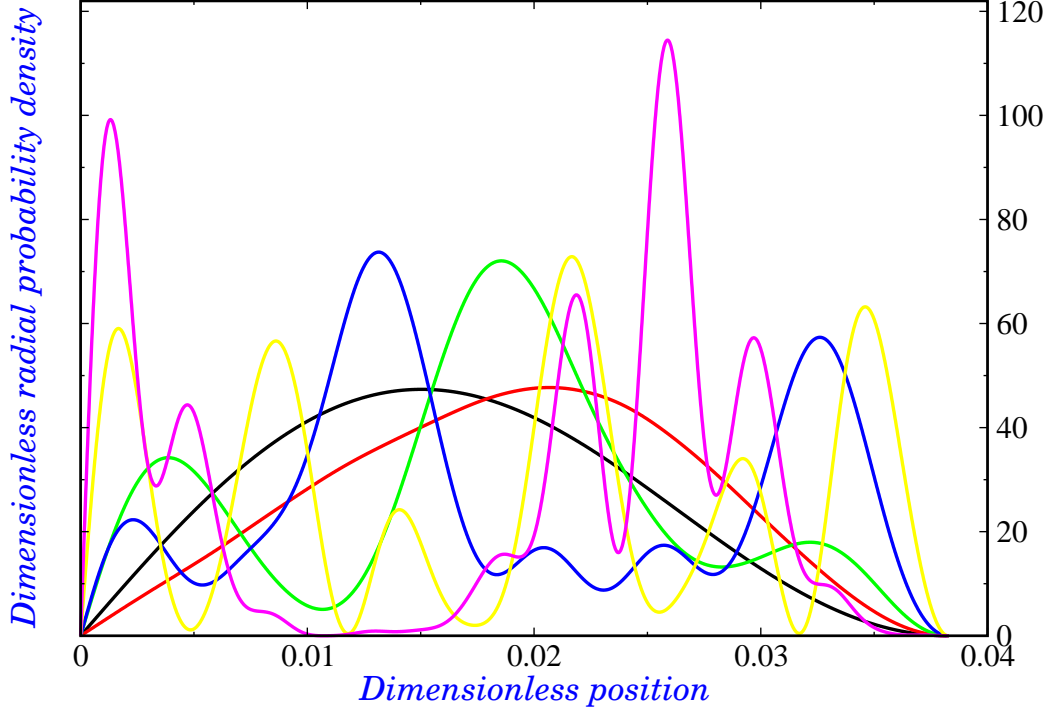


FIG. 4: (Color online) Dimensionless radial probability density $\varrho_{0,1}(\eta, T)$ against dimensionless position coordinate η at the instant of time when the wall arrives at $\xi = 0.1$, for six different values of contraction rate; $\alpha = -0.01 \alpha_{0,1}$ (black curve), $\alpha = -\alpha_{0,1}$ (red curve), $\alpha = -5 \alpha_{0,1}$ (green curve), $\alpha = -10 \alpha_{0,1}$ (blue curve), $\alpha = -15 \alpha_{0,1}$ (yellow curve) and $\alpha = -20 \alpha_{0,1}$ (magenta curve). For a rapidly moving boundary, the sudden approximation does not work for a contracting box, but, in the opposite limit the process is adiabatic.

$$\begin{aligned}
C_{mnn}^{(1)} &= - \int_0^1 ds \, s \left(\frac{dJ_m(x_{mn}s)}{ds} \right)^2 \\
&= - \frac{(x_{mn})^2}{4} \left[- \frac{(x_{mn})^{2m} {}_3F_4\left(m + \frac{1}{2}, m+1, m+1; m, m+2, m+2, 2m+1; -(x_{mn})^2\right)}{4^m (m-1)! \, m!} \right. \\
&\quad + \frac{2J_{m-2}(x_{mn})J_{m-1}(x_{mn}) (4m(m^2-1) + (x_{mn})^2(1-2m))}{(x_{mn})^3} + (J_{m-2}(x_{mn}))^2 \frac{(x_{mn})^2 - 2m(m+1)}{(x_{mn})^2} \\
&\quad \left. + (J_{m-1}(x_{mn}))^2 \left(\frac{1}{2} - \frac{4m(m-1)(2m^2-2-(x_{mn})^2)}{(x_{mn})^4} \right) + \frac{(J_{m+1}(x_{mn}))^2}{2} \right] \quad \text{if } m \neq 0 \quad (41)
\end{aligned}$$

and,

$$C_{mnn}^{(1)} = -\frac{1}{2} x_{0n} (J_1(x_{0n}))^2 \quad \text{if } m = 0, \quad (42)$$

where the generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ in terms of the Pochhammer symbol [13],

$$(a)_n = \frac{(a+n-1)!}{(a-1)!}, \quad (a)_0 = 1,$$

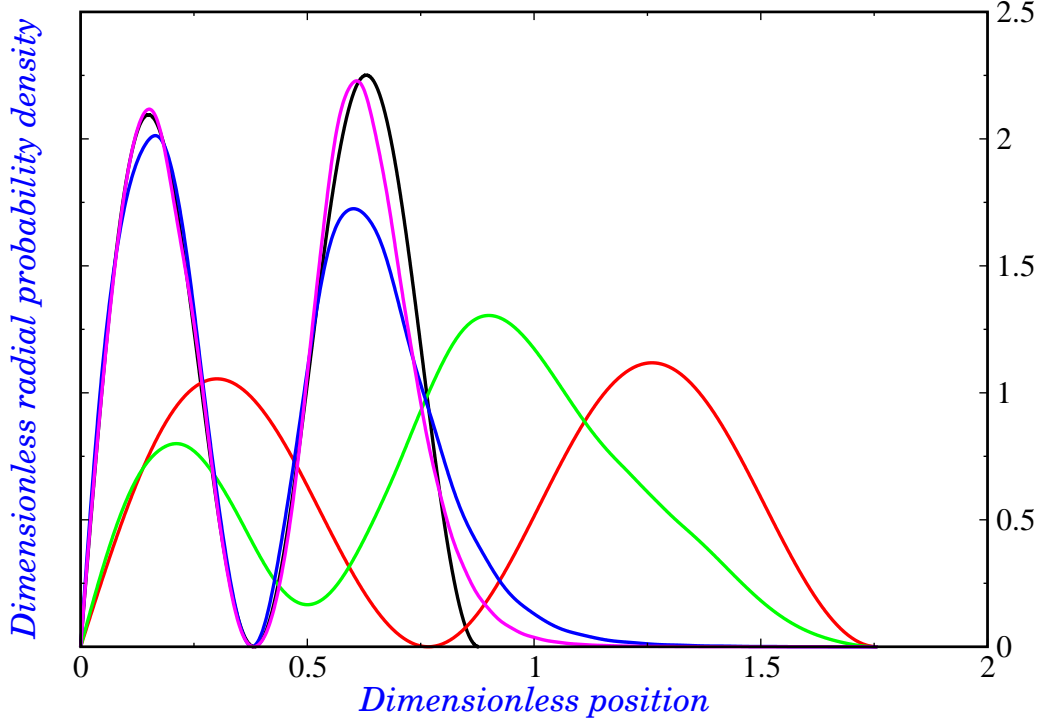


FIG. 5: (Color online) Dimensionless radial probability density $\varrho_{0,2}(\eta, T)$ against dimensionless position coordinate η at the instant of time when the wall arrives at $\xi = 2$, for five different values of expansion rate; $\alpha = 0$ (black curve), $\alpha = 0.01 \alpha_{0,2}$ (red curve), $\alpha = \alpha_{0,2}$ (green curve), $\alpha = 5 \alpha_{0,2}$ (blue curve) and $\alpha = 10 \alpha_{0,2}$ (magenta curve). Both the adiabatic and the sudden approximation work for an expanding box.

becomes

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!},$$

and the regularized generalized hypergeometric function ${}_p\bar{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ is

$${}_p\bar{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \frac{{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)}{\Gamma(b_1) \dots \Gamma(b_q)},$$

and $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the Gamma function. Recurrence relations of the Bessel functions have been used in (39) to re-write the integrand, which is more appropriate for numerical calculations.

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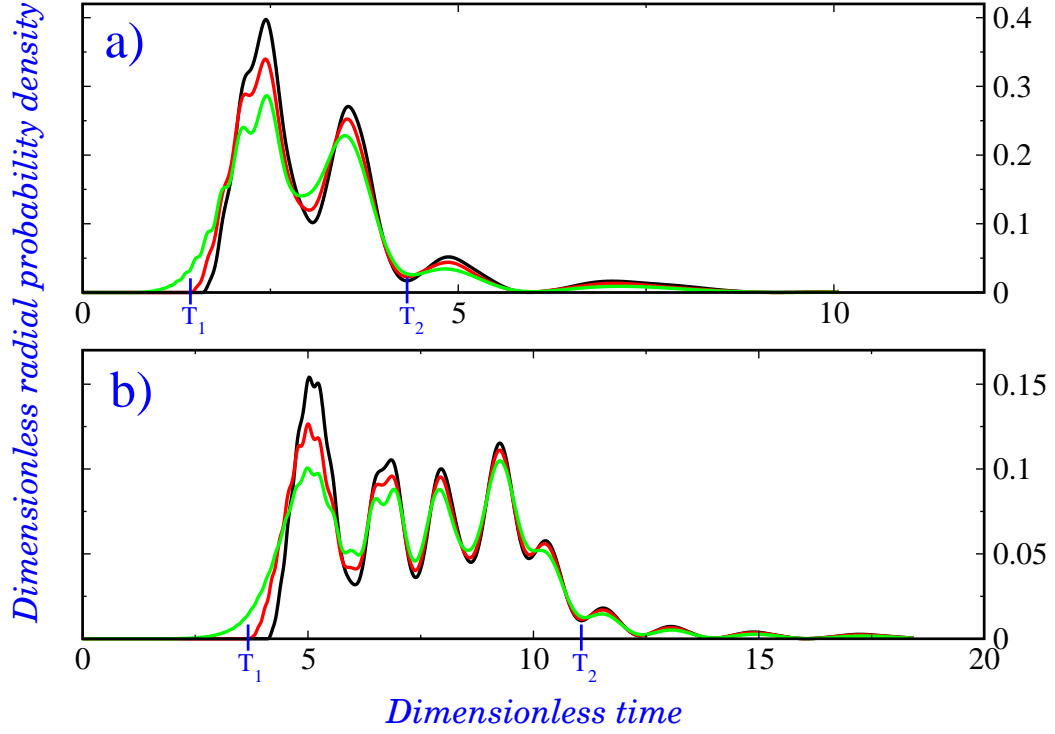


FIG. 6: (Color online) Dimensionless radial probability density $\varrho_{mn}(\eta, T)$ for a particle initially in the state (a) $u_{0,6}$ and (b) $u_{0,15}$, against dimensionless time coordinate T at dimensionless observation point $\eta^{(0)} = x_{mn}/\pi$, for three different values of velocity parameter; $\alpha = 0.9 \alpha_{mn}$ (black curve), $\alpha = \alpha_{mn}$ (red curve), $\alpha = 2 \alpha_{mn}$ (green curve). T_1 and T_2 are dimensionless classical flight times from the front and back edges of the circle to the dimensionless observation point. As the velocity of the moving boundary increases, the fringe visibility reduces.

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